Appendix 13: Hyperbolic Functions

A13.1. General

Exponential, trigonometric and hyperbolic functions are all solutions to the following differential equation:

$$\frac{d^2y}{dx^2} = ay \text{ , with } a^2 = 1 \qquad \text{(A13.1.1)}$$

Properties of these functions and relations between them are provided in Table A13.1:

Exponential Function	**Trigonometric Functions**	**Hyperbolic Functions**
Solution to: $y' = y$ ($a = 1$)	Solution to: $y'' = -y$ ($a = -1$)	Solution to: $y'' = y$ ($a = 1$)
$e^0 = 1$	$\cos 0 = 1$ $\sin 0 = 0$	$\cosh 0 = 1$ $\sinh 0 = 0$
$e^x = \cosh x + \sinh x$	$\cos x = \frac{e^{ix} + e^{-ix}}{2}$ $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$	$\cosh x = \frac{e^{x} + e^{-x}}{2}$ $\sinh x = \frac{e^{x} - e^{-x}}{2}$
	$\tan x = \frac{\sin x}{\cos x}$	$\tanh x = \frac{\sinh x}{\cosh x}$
	$\cot x = \frac{\cos x}{\sin x}$	$\coth x = \frac{\cosh x}{\sinh x}$
	$\cos^2 x + \sin^2 x = 1$	$\cosh^2 x - \sinh^2 x = 1$
$\frac{de^x}{dx} = e^x$	$\frac{d\cos x}{dx} = -\sin x$ $\frac{d\sin x}{dx} = \cos x$	$\frac{d\cosh x}{dx} = \sinh x$ $\frac{d\sinh x}{dx} = \cosh x$
$e^{ix} = \cos x + i \sin x$	$\cos ix = \cosh x$ $\sin ix = i \sinh x$	$\cosh ix = \cos x$ $\sinh ix = i \sin x$

Table A13.1 Properties of and relations between the exponential function, trigonometric functions and hyperbolic functions

A13.2. Series Expansions

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \tag{A13.2.1}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots \tag{A13.2.2}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots \tag{A13.2.3}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots \tag{A13.2.4}$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots \tag{A13.2.5}$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} \ldots \quad |x| < \pi/2 \tag{A13.2.6}$$

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} + \ldots 0 < |x| < \pi \tag{A13.2.7}$$

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} \ldots \quad |x| < \pi/2 \tag{A13.2.8}$$

$$\coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} + \ldots 0 < |x| < \pi \tag{A13.2.9}$$