Appendix 14: Stirling Approximation

Factorials can be approximated for large values of n using the Stirling approximation which is given by:

$$n! = \sqrt{2\pi n}\, n^n \exp[-n]\{1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840\, n^3} - \frac{571}{2488320\, n^5} + O(\frac{1}{n^5})\} \qquad \text{(A14.1)}$$

An alternate form, which is of particular interest if the logarithm of $n!$ must be calculated is given by:

$$n! = \sqrt{2\pi n}\, n^n \exp[-n + \frac{1}{12n} + O(\frac{1}{n^2})] \qquad \text{(A14.2)}$$

The exact and approximate values for n = 1, 2, ... 20 and the relative error as calculated using (A14.1) and (A14.2) are provided in the Table A14.1.

n	*n!*	Equation (A14.1)	Relative Error	Equation (A14.2)	Relative Error
1	1	1.00	5.01E-04	1.00	2.27E-03
2	2	2.00	2.10E-05	2.00	3.26E-04
3	6	6.00	3.00E-06	6.00	9.99E-05
4	24	24.00	7.33E-07	24.00	4.27E-05
5	120	120.00	2.44E-07	120.00	2.20E-05
6	720	720.00	9.89E-08	720.01	1.28E-05
7	5040	5040.00	4.60E-08	5040.04	8.05E-06
8	40320	40320.00	2.37E-08	40320.22	5.40E-06
9	362880	362880.00	1.32E-08	362881.38	3.80E-06
10	3628800	3628799.97	7.79E-09	3628810.05	2.77E-06
11	39916800	39916799.81	4.84E-09	39916883.11	2.08E-06
12	479001600	479001598.50	3.14E-09	479002368.48	1.60E-06
13	6227020800	6227020786.90	2.10E-09	6227028659.89	1.26E-06
14	87178291200	87178291073.35	1.45E-09	87178379323.32	1.01E-06
15	1307674368000	1307674366653.88	1.03E-09	1307675442913.47	8.22E-07
16	20922789888000	20922789872396.30	7.46E-10	20922804061389.80	6.77E-07
17	355687428096000	355687427900040.00	5.51E-10	355687629001078.00	5.65E-07
18	6402373705728000	6402373703076830.00	4.14E-10	6402376752492220.00	4.76E-07
19	121645100408832000	121645100370383000.00	3.16E-10	121645149634119000.00	4.05E-07
20	2432902008176640000	2432902007581510000.00	2.45E-10	2432902852332160000.00	3.47E-07

Table A14.1 Value and relative accuracy of $n!$ as calculated using (A14.1) and (A14.2)

It is of interest to further examine the accuracy as a function of the number of terms. Figure A14.1 presents the relative error for n ranging from 1 to 100 as calculated using the first, 2, 3, 4 and 5 terms of (A14.1) as well as equation (A14.2).

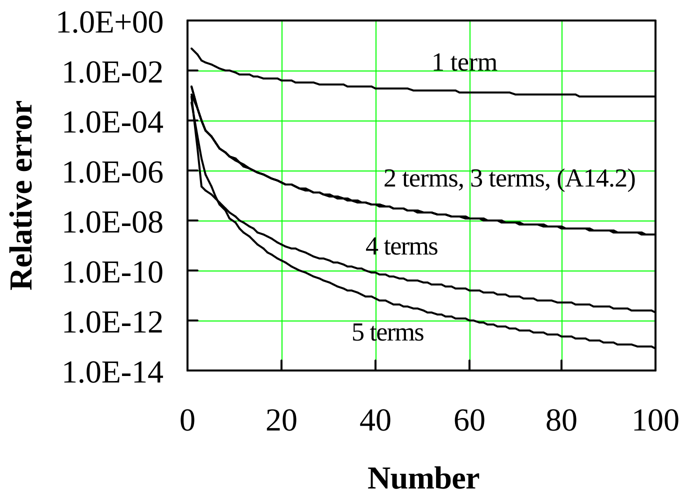

Figure A14.1 Relative accuracy of $n!$ as a function of n. Compared are the accuracy when using 1, 2, 3, 4, or 5 terms of (A14.1) and (A14.2)

From the figure one finds that there is a significant improvement in accuracy between using only the first term versus using the first two terms of the approximation. The first term results in an accuracy better than 1%, while two or more terms provide an accuracy better than 10^{-6} for integers larger than 15. Surprisingly there is little improvement when adding the third term. Adding the fourth and fifth term does further improve the accuracy. For most applications one will find that two terms of (A14.1) or (A14.2) will provide sufficient accuracy.

If the natural logarithm of $n!$ is of interest it can be obtained from (A14.2), yielding:

$$\ln n! = \ln \sqrt{2p} + (n + \frac{1}{2}) \ln n - n + \frac{1}{12n} + O(\frac{1}{n^2}) \qquad \text{(A14.3)}$$