A15. Appendix 15: Optics

A15.1. Transmission and reflection at dielectric interfaces

The transmission and reflection of an electromagnetic wave at a dielectric interface with refractive index n_0 and n_1, can be obtained by requiring both continuity of the field at the interface (this assumes a transverse polarized wave which is incident normal to the interface) and conservation of power flow:

$$P = \frac{c}{\sqrt{m_r e_r}} e_r E^2 = cnE^2 \qquad \text{(A15.1.1)}$$

This leads to the following relations between the incident, reflected and transmitted field components (all in the plane of the interface)[1]:

$$E_I - E_R = E_T \text{ or } 1 - r = t \qquad \text{(A15.1.2)}$$

$$n_0 E_I^2 = n_0 E_R^2 + n_1 E_T^2 \text{ or } 1 - r^2 = \frac{n_1}{n_0} t^2 \qquad \text{(A15.1.3)}$$

Where the reflection and transmission coefficients, r and t are given by:

$$r = \frac{E_R}{E_I} \text{ and } t = \frac{E_T}{E_I} \qquad \text{(A15.1.4)}$$

The individual electric field components are shown in Figure A15.1. The incident wave is assumed to travel from left to right.

[1]Equation (A15.1.2) also applies also to multi-layer structures without gain or absorption. It can be shown that the following relations exist between $r = r_{01}$, $r' = r_{10}$, $t = t_{01}$ and $t' = t_{10}$: $r = - r'$ and $1 = tt' - rr'$

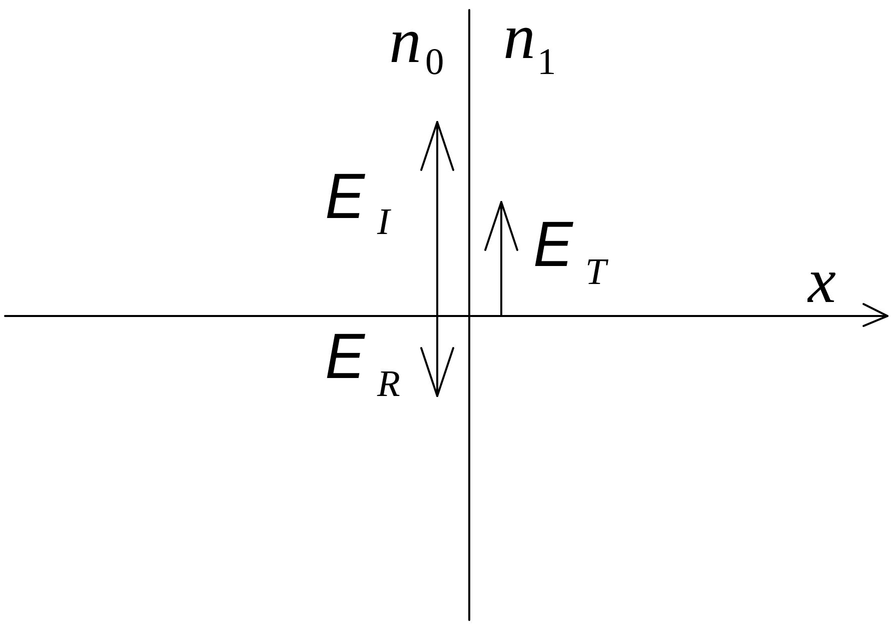

Figure A15.1: Incident, reflected and transmitted wave at a dielectric interface.

These equations can be solved yielding the transmission and reflection coefficients, t_{01} and r_{01}.

$$t = t_{01} = \frac{E_T}{E_I} = \frac{2n_1}{n_0 + n_1} \quad \text{(A15.1.5)}$$

And

$$r = r_{01} = 1 - t_{01} = \frac{E_R}{E_I} = \frac{n_0 - n_1}{n_0 + n_1} \quad \text{(A15.1.6)}$$

where the subscripts 0 and 1 are used to indicate that the incident wave travels from the material with index n_0 towards the material with index n_1. The fraction of the power reflected, R, and transmitted, T, are given by:

$$R = \frac{P_R}{P_I} = \frac{n_0 E_R^2}{n_0 E_I^2} = \frac{(n_0 - n_1)^2}{(n_0 + n_1)^2} \quad \text{(A15.1.7)}$$

$$T = \frac{P_T}{P_I} = \frac{n_1 E_T^2}{n_0 E_I^2} = \frac{4 n_0 n_1}{(n_0 + n_1)^2} \qquad \text{(A15.1.8)}$$

This result can be extended for any angle of incidence f_0, yielding

$$r_{01,TE} = \frac{n_0 \cos f_0 - n_1 \cos f_1}{n_0 \cos f_0 + n_1 \cos f_1} \text{ and } R_{01,TE} = \left| \frac{n_0 \cos f_0 - n_1 \cos f_1}{n_0 \cos f_0 + n_1 \cos f_1} \right|^2 \qquad \text{(A15.1.9)}$$

$$r_{01,TM} = \frac{n_q \cos f_0 - n_0 \cos f_1}{n_1 \cos f_0 + n_0 \cos f_1} \text{ and } R_{01,TM} = \left| \frac{n_1 \cos f_0 - n_0 \cos f_1}{n_1 \cos f_0 + n_0 \cos f_1} \right|^2 \qquad \text{(A15.1.10)}$$

$$R = \frac{P_R}{P_I} = \frac{n_0 E_T}{n_0 EE} = \frac{2n_1}{n_0 + n_1} \text{ and } r = r_{01} = 1 - t_{01} = \frac{E_R}{E_I} = \frac{n_0 - n_1}{n_0 + n_1} \qquad \text{(A15.1.11)}$$

with

$$n_1 \sin f_1 = n_0 \sin f_0 \qquad \text{(A15.1.12)}$$

where f_0 is the angle of the transmitted wave with respect to the normal to the interface as shown in the figure below:

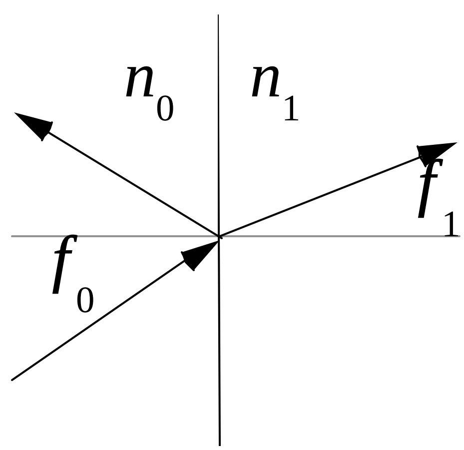

Figure A15.2: Incident, reflected and transmitted wave at a dielectric interface.

R_{TE} is the reflectivity if the electric field is parallel to the interface while R_{TM} is the reflectivity if the magnetic field is parallel to the interface. An example of the reflection as a function of the incident angle is shown in Figure A15.3.

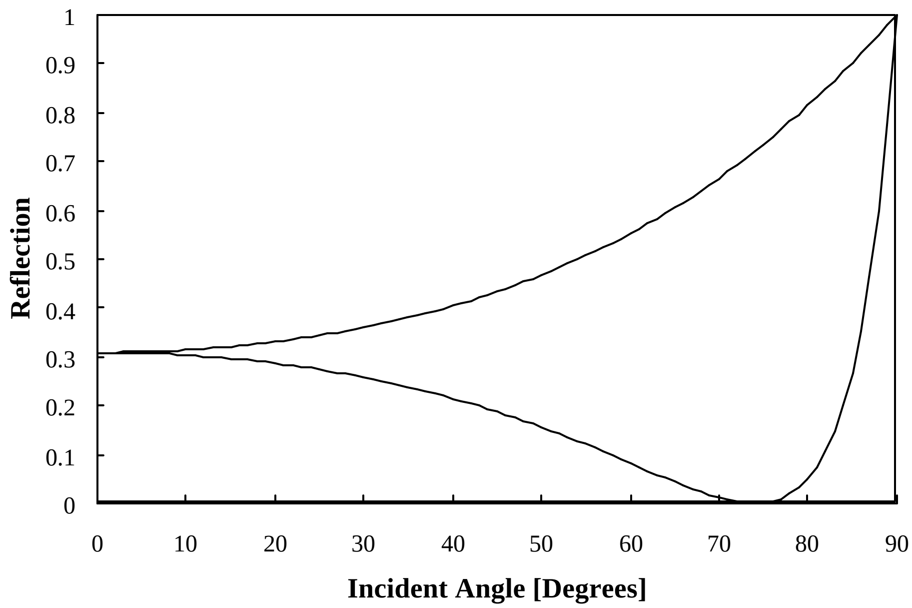

Figure A15.3 Reflectivity at an air-GaAs interface (n_0 = 1, n_1 = 3.5) as a function of the incident angle for both polarizations (TE and TM) of the incident wave. The reflectivity, R_{TM} goes to zero if the incident angle equals $\tan^{-1}$ (n_1/n_0)

A15.2. Transmission and reflection of a multi-layer dielectric structure

In order to calculate the reflection and transmission of a multi-layer structure, we first define the electric field components left and right of the first interface: two field components corresponding to the forward propagating waves, $E_{0,f}$ and $E'_{1,f}$ and two field components corresponding to the backward propagating waves, $E_{0,b}$ and $E'_{1,b}$. All field components are assumed to be parallel to the plane of the interface.

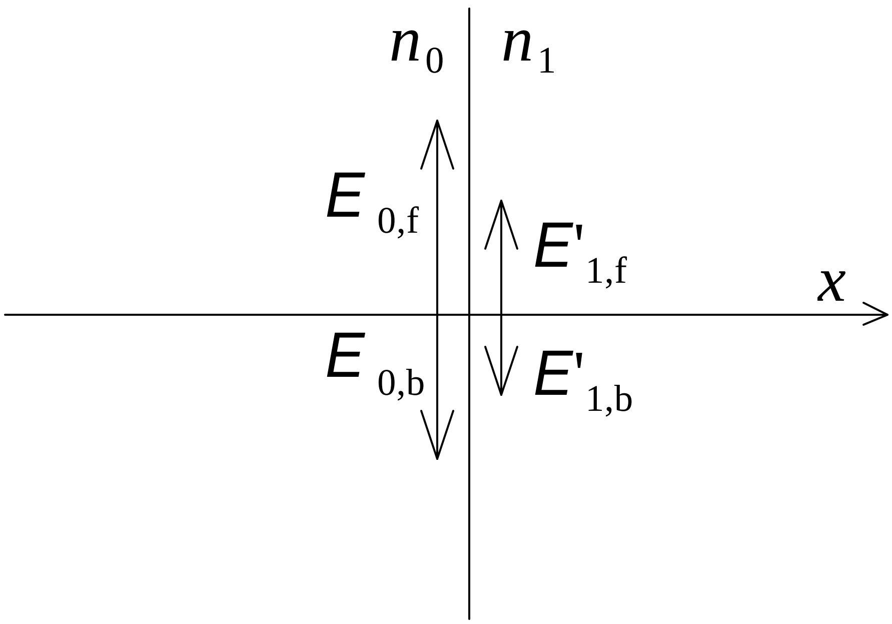

Figure A15.4 Forward and backward propagating waves at a dielectric interface.

Using the result for a single interface and applying superposition in the case of two waves incident on the dielectric interface, one from each side, one finds the following relation between the field components:

$$E'_{1,f} = t_{10}E_{0,f} + r_{01}E'_{1,b} \tag{A15.2.1}$$

$$E_{0,b} = t_{10}E'_{1,b} + r_{01}E'_{0,f} \tag{A15.2.2}$$

from which a relation can be obtained between the field components, just before the next interface:

$$\begin{pmatrix} E_{1,f} \\ E_{1,b} \end{pmatrix} = \Phi_1 A_{01} \begin{pmatrix} E_{0,f} \\ E_{0,b} \end{pmatrix} \tag{A15.2.3}$$

with

$$A_{01} = \frac{1}{2n_1}\begin{pmatrix} n_1+n_0 & n_1-n_0 \\ n_1-n_0 & n_1+n_0 \end{pmatrix} \text{ and } \Phi_1 = \begin{pmatrix} e^{ik_1d_1} & 0 \\ 0 & e^{-ik_1d_1} \end{pmatrix} \quad \text{(A15.2.4)}$$

The matrix Φ_1 accounts for the phase shift through the layer with index n_1 and thickness d_1. Note that that $A_{01}A_{10}$ should equal a unity matrix so that $A_{10}^{-1} = A_{01}$.

This result can easily be extended to a set of N interfaces for which

$$\begin{pmatrix} E_{N,f} \\ E_{N,b} \end{pmatrix} = \Phi_N A_{N-1,N} \Phi_{N-1} A_{N-2,N-1} \ldots \Phi_2 A_{12} \Phi_1 A_{01} \begin{pmatrix} E_{0,f} \\ E_{0,b} \end{pmatrix} \quad \text{(A15.2.5)}$$

15.2.1. Example: a Distributed Bragg Reflector (DBR)

Consider a DBR structure consisting of N periods of two quarter-wavelength layers with alternating index n_1 and n_2, between layers with index n_0 and n_3. The relation between the field components just after the last interface and those just before the first interface is given by:

$$\begin{pmatrix} E_f \\ E_b \end{pmatrix} = A_{23} A_{12} [A_{21} \Phi_2 A_{12} \Phi_1]^N A_{01} \begin{pmatrix} E_{0,f} \\ E_{0,b} \end{pmatrix} \quad \text{(A15.2.6)}$$

The subscripts refer to the index of refraction left and right of the interface. The matrix A_{12} was added to cancel the last matrix A_{21} of the N^{th} product. The N^{th} power of the matrices characterizing one period of the DBR is given by:

$$[A_{21}\Phi_2 A_{12}\Phi_1]^N = \frac{(-1)^N}{2}\begin{pmatrix} (\frac{n_1}{n_2})^N + (\frac{n_1}{n_2})^{-N} & (\frac{n_1}{n_2})^N - (\frac{n_1}{n_2})^{-N} \\ (\frac{n_1}{n_2})^N - (\frac{n_1}{n_2})^{-N} & (\frac{n_1}{n_2})^N + (\frac{n_1}{n_2})^{-N} \end{pmatrix} \quad \text{(A15.2.7)}$$

assuming Φ_1 and Φ_2 to equal $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, as is the case for quarter-wavelength layers, yielding:

$$\begin{pmatrix} E_f \\ E_b \end{pmatrix} = \frac{(-1)^N}{2}\begin{pmatrix} \frac{n_3}{n_0}(\frac{n_1}{n_2})^N + (\frac{n_1}{n_2})^{-N} & \frac{n_3}{n_0}(\frac{n_1}{n_2})^N - (\frac{n_1}{n_2})^{-N} \\ \frac{n_3}{n_0}(\frac{n_1}{n_2})^N - (\frac{n_1}{n_2})^{-N} & \frac{n_3}{n_0}(\frac{n_1}{n_2})^N + (\frac{n_1}{n_2})^{-N} \end{pmatrix}\begin{pmatrix} E_{0,f} \\ E_{0,b} \end{pmatrix} \quad \text{(A15.2.8)}$$

The total reflectivity of the structure can be obtained by requiring E_b to be zero so that

$$R = \frac{P_R}{P_I} = \frac{|E_{0b}|^2}{|E_{0f}|^2} = \left(\frac{1 - \frac{n_3}{n_0}(\frac{n_1}{n_2})^{2N}}{1 + \frac{n_3}{n_0}(\frac{n_1}{n_2})^{2N}} \right)^2 \tag{A15.2.9}$$

15.2.2. Spreadsheet solution to an arbitrary layer structure

The matrix for an arbitrary layer with index n_1 and thickness d_1 following a layer with index n_0 can be written as a function of the real and imaginary components of the fields:

$$\mathrm{Re}(E_{1f}) = \cos\phi_1\,(a_{11}\mathrm{Re}(E_{0f}) + a_{12}\mathrm{Re}(E_{0b})) \tag{A15.2.10}$$

$$-\sin\phi_1\,(a_{11}\mathrm{Im}(E_{0f}) + a_{12}\mathrm{Im}(E_{0b}))$$

$$\mathrm{Im}(E_{1f}) = \sin\phi_1\,(a_{11}\mathrm{Re}(E_{0f}) + a_{12}\mathrm{Re}(E_{0b}))$$

$$+ \cos\phi_1\,(a_{11}\mathrm{Im}(E_{0f}) + a_{12}\mathrm{Im}(E_{0b}))$$

$$\mathrm{Re}(E_{1b}) = \cos\phi_1\,(a_{12}\mathrm{Re}(E_{0f}) + a_{11}\mathrm{Re}(E_{0b}))$$

$$+ \sin\phi_1\,(a_{12}\mathrm{Im}(E_{0f}) + a_{11}\mathrm{Im}(E_{0b}))$$

$$\mathrm{Im}(E_{1b}) = -\sin\phi_1\,(a_{12}\mathrm{Re}(E_{0f}) + a_{11}\mathrm{Re}(E_{0b}))$$

$$+ \cos\phi_1\,(a_{12}\mathrm{Im}(E_{0f}) + a_{11}\mathrm{Im}(E_{0b}))$$

with $\phi_1 = k_1 d_1$ and $a_{11} = \frac{n_1 + n_0}{2n_1}$ $a_{12} = \frac{n_1 - n_0}{2n_1}$

This set of equations can also be interpreted as a recursion relation between the fields at the end of the current layer and the fields at the end of the previous layer. The field within a given layer is obtained by dividing this layer into different section with the same index of refraction and a combined thickness, which equals the layer thickness. Implementation into a spreadsheet provides the actual fields throughout the structure. An example is shown in Figure A15.5:

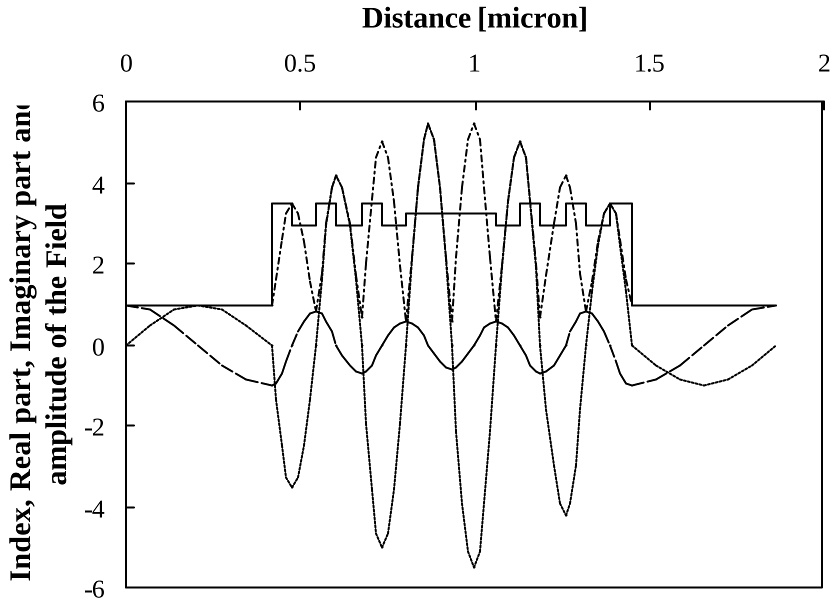

Figure A15.5 Refractive index as well as real part, imaginary part and amplitude of the total field due to the forward and backward propagating waves. The incident wave is a forward propagating wave with unity amplitude and zero phase at $x = 0$

15.2.3. Reflection and transmission through multiple layers

The calculation of the reflection and transmission through multiple layers can in general be obtained by using equation (A15.2.5), assuming that no incident wave is present after the *N*th interface or $E_{Nb} = 0$, as was used to calculate the reflection of a distributed Bragg reflector in section A15.2.1 For the general case this requires inverting the matrix linking the incident fields E_{0f} and E_{0b} to the fields on the other side of the structure, E_{Nf} and E_{Nb}. This procedure can be avoided by using the time reversal principle, which applies in the absence of magnetic fields, so that setting the incident fields equal to:

$$E_{0f} = 1 \text{ and } E_{0b} = 0 \tag{A15.2.11}$$

one finds the reflection, *R*, and transmission, *T*, through the structure from:

$$R = \frac{|E_{Nb}|^2}{|E_{Nf}|^2} \text{ and } T = \frac{n_N}{n_0} \frac{1}{|E_{Nf}|^2} \tag{A15.2.12}$$

These expressions can even be applied to the case where some or even all the layers are absorbing or have gain, by using the complex conjugate of the refractive indices: As the waves travel through the structure in the reverse direction, one finds that absorbing regions provide gain while regions with gain become absorbing. Inverting the sign of the imaginary part of the refractive index for each region therefore provides the correct absorption/gain when reversing time. An example for the case of a thin silver layer is shown in Figure A15.6.

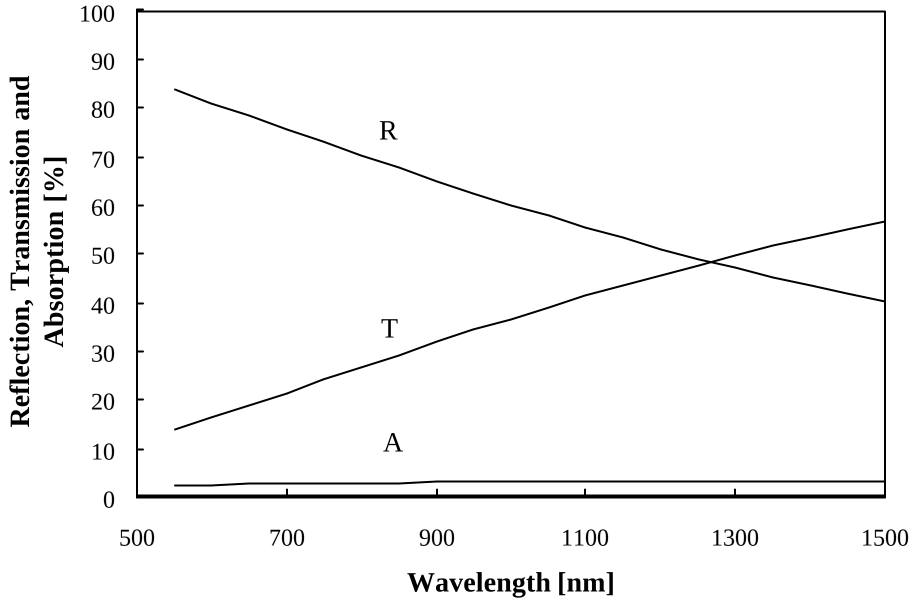

Figure A15.6 Reflection, *R*, transmission, *T*, and absorption, *A*, versus wavelength of a 10 nm silver layer in air. The refractive index of the silver was assumed to be n = 0.102 + i 6.22.

A15.3. Fabry-Perot cavity

Consider a structure consisting of two reflecting interfaces with reflection coefficients r_{01} and r_{12}, transmission coefficients t_{01} and t_{12}. The two reflecting surfaces are separated by a medium with thickness d, refractive index n and absorption coefficient a. The reflection and transmission amplitudes are given by[2]:

$$\frac{A_R}{A_I} = \frac{r_{01} + r_{12} e^{-id}}{1 + r_{01} r_{12} e^{-id}} \; , \; \frac{A_T}{A_I} = \frac{t_{01} t_{12} e^{-id/2}}{1 + r_{01} r_{12} e^{-id}} \qquad \text{(A15.3.1)}$$

[2]See Yariv, "Optical Electronics", Fourth edition, Holt Reinhart and Winston, Inc, 1991, p 112-115.

For a symmetric structure with $r_{01} = - r_{12}$ and $t_{01} = t_{12}$ one obtains the following reflection, transmission and absorption intensities.

$$\text{Reflection} = \frac{I_R}{I_I} = \frac{A_R A_R^*}{A_I A_I^*} = \frac{R(1-2A\cos\delta + A^2)}{1-2RA\cos\delta + R^2A^2} \qquad \text{(A15.3.2)}$$

$$\text{Transmission} = \frac{I_T}{I_I} = \frac{A_T A_T^*}{A_I A_I^*} = \frac{A(1-R)^2}{1-2RA\cos\delta + R^2A^2} \qquad \text{(A15.3.3)}$$

$$\text{Absorption} = \frac{A(2\cos\delta\,(1-R) + AR^2 - (1-R)^2)}{1-2RA\cos\delta + R^2A^2} \qquad \text{(A15.3.4)}$$

with

$$\delta = \frac{4\pi\, nd\cos\phi_0}{\lambda} - i\frac{\alpha\, d}{\cos\phi_0}\;,\; A = \mathrm{e}^{-\alpha d/\cos\phi_0} \qquad \text{(A15.3.5)}$$

These equations enable to calculate reflection, transmission and absorption as a function of cavity length or as a function of photon energy, E_{ph} [eV] = $\frac{hc}{q\lambda}$. Examples are shown in the following figures:

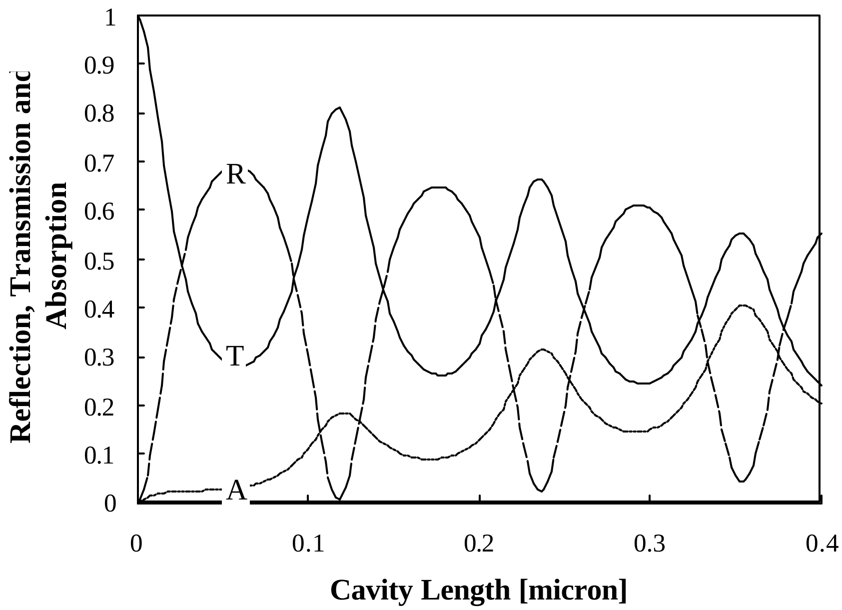

Figure A15.7 Reflection, Transmission and Absorption versus cavity length of a GaAs Fabry-Perot cavity at 830 nm under normal incidence.

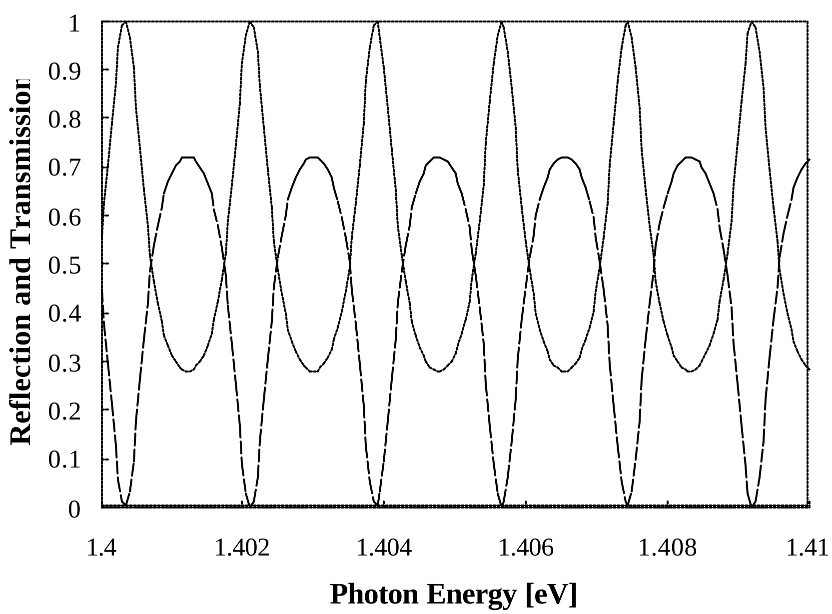

Figure A15.8 Reflection and transmission versus photon energy of a 100 μm GaAs Fabry-Perot cavity under transparency conditions (i.e. α = 0)

A15.4. Ellipsometer Equations

An ellipsometer enables to measure the refractive index and the thickness of semi-transparent thin films. The instrument relies on the fact that reflection at a dielectric interface depends on the polarization of the light. It consists of a laser whose state of polarization can be modified with a polarizer. The beam is reflected off the layer of interest and then analyzed with a polarizer. The operator changes the angle of the polarizer and analyzer until a minimal signal is detected. The angles are then related to the reflections for both polarizations in the following way:

$$\frac{r_{TM}}{r_{TE}} = \tan \Psi e^{i\Delta} \tag{A15.4.1}$$

The angles Ψ and Δ are related to the measured angles, P_1, A_1, P_2 and A_2 in the following way:

$$\Delta = \frac{3p}{2} - 2P_1 = \frac{5p}{2} - 2P_2 = 2p - P_1 - P_2 \tag{A15.4.2}$$

$$\Psi = A_1 = p - A_2 = \frac{A_1 - A_2 + p}{2} \tag{A15.4.3}$$

The minimal signal is obtained when both polarization incident on the analyzer are in phase. This can be obtained for two different positions of the polarizer, hence the two values P_1 and P_2. In principle one could measure either one. In practice both values are measured to eliminate any possible misalignment of the instrument thereby yielding a more accurate result.

A theoretical analysis of the $\Psi - \Delta$ curves is obtained by combining the expressions for the reflectivities at both dielectric interfaces (A15.1.9) and (A15.1.10) with the expression for the asymmetric Fabry-Perot cavity (A15.3.1). An example of such curves as obtained for silicon dioxide layers (n_1 = 1.455) on silicon (n_2 = 3.875 – i 0.018) using a helium-neon laser (λ = 0.6328 μm) is in Figure A15.9.

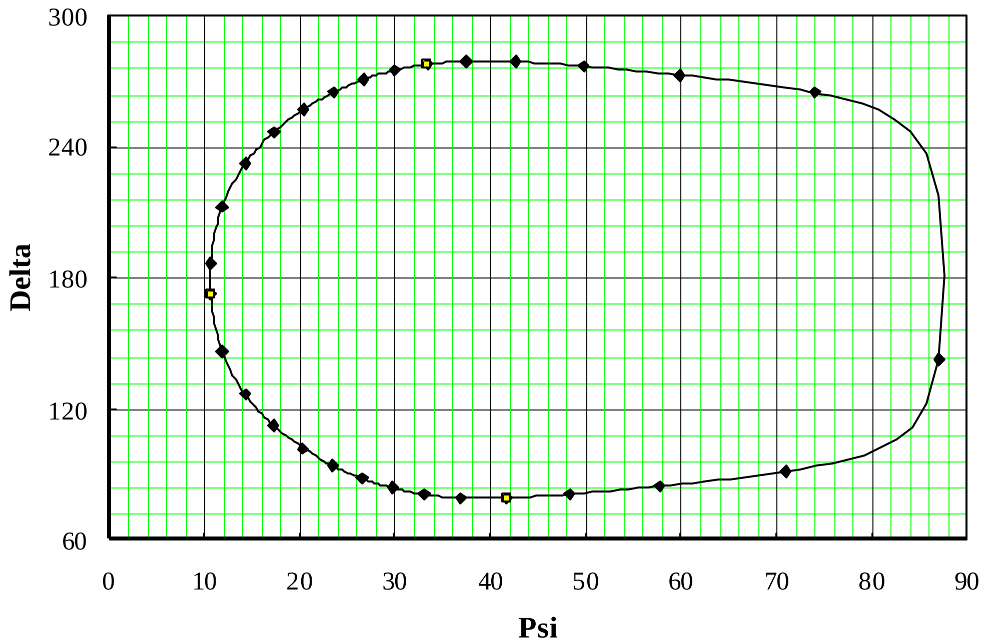

Figure A15.9 Ψ-Δ curves for silicon dioxide on silicon. Thickness increases counter clock wise from 0 (square marker on the left) in steps of 10 nm (black diamonds) and in steps of 100 nm (squares). Incident angle of the laser beam is 70 degrees.

Since the silicon dioxide was assumed to be transparent one finds the values for both Ψ and Δ to be identical for layers which differ in thickness by $\frac{l}{2n_1 \cos f_1}$ = 0.284 µm. The corresponding curves for the measured values P_1, A_1, P_2 and A_2 are also shown in Figure A15.10:

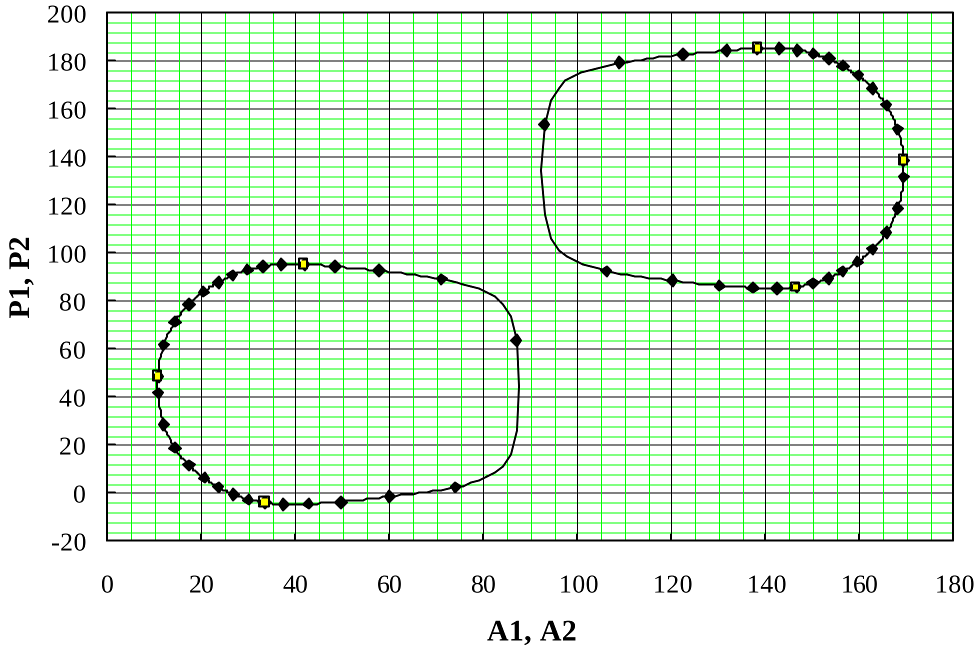

Figure A15.10 A_1-P_1 and A_2-P_2 curves for silicon dioxide on silicon. Thickness increases counter clock wise from 0 (at the square marker on the left) for A_1 versus P_1 and counter clock wise from 0 (square marker on the right) for A_2 versus P_2, both in steps of 10 nm (black diamonds) and in steps of 100nm (squares). Incident angle of the laser beam is 70 degrees from the normal to the surface.

A15.5. Interference colors of thin transparent films

Transparent thin films often have strikingly bright colors, which vary rapidly with thickness. These colors are caused by interference of the light reflected at the front and back interface of the film and vary depending on the refractive index of the film as well as that of the substrate. The analysis is simply based on the general expressions for the Fabry-Perot cavity (A15.3.1), which can be further manipulated to yield the total reflectivity as a function of the wavelength.

$$R(l)n = \frac{A_R A_R^*}{A_I A_I^*} = \frac{r_{01}^2 + 2r_{01}r_{12}\cos d + r_{12}^2}{1 + 2r_{01}r_{12}\cos d + r_{01}^2 r_{12}^2} \qquad \text{(A15.5.1)}$$

With

$$d = \frac{4p\, n_1 d \cos f_1}{l} \qquad \text{(A15.5.2)}$$

The reflection can then be calculated as a function of the wavelength corresponding to the primary colors red, green and blue. A useful way to eliminate one of the primary colors in the determination of the observed color is by determining the intensity of white light as an equal amount of red, green and blue light. The actual colors observed for each thickness depend on the relative intensities of the remaining primary colors. An example is shown in the figure below for silicon dioxide on a silicon substrate.

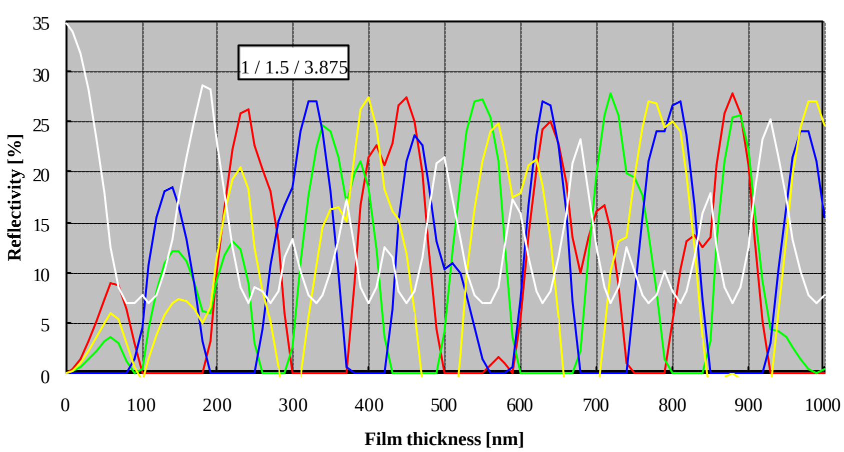

Figure A15.11 Relative intensities of the colors reflected off a silicon dioxide layer on a silicon substrate.

Table A15.1 lists the color as a function of the thickness for some different materials.

	air/ water/ air n_0 / n_1 / n_2 1 / 1.33 / 1		air / water / glass n_0 / n_1 / n_2 1 / 1.33 / 1.5
< 10 nm	black	< 10 nm	White
100nm	white	100 nm	Black
150 nm	brownish-yellow	150 nm	Blue
200 nm	violet	210 nm	White
250 nm	blue	230 nm	Yellow
300 nm	light green	270 nm	Red
330 nm	orange	360 nm	Blue
400 nm	pink	400 nm	Green
450 nm	light blue	450 nm	Yellow
500 nm	dark green	500 nm	Red
600 nm	red	530 nm	Pink
650 nm	turquoise	610 nm	Green

Table A15.1: Expected color versus thickness for thin layers of water in air or on a glass surface